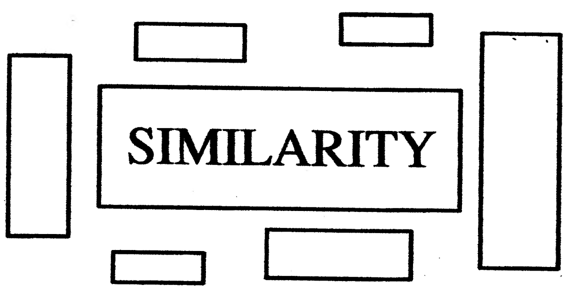
Project MATHEMATICS!

Program Guide and Workbook

to accompany the videotape on



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Program Guide and Workbook

to accompany the videotape on

SIMILARITY

Written by TOM M. APOSTOL, California Institute of Technology

with the assistance of the

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SIMILARITY

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! uses computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes the important points in the capsule. Some sections contain exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief *Review of prerequisites* dealing with ratios, a concept the student should become familiar with before viewing the tape. The program introduces the concept of similarity as the intuitive idea of objects having the same shape but not necessarily the same size. After showing many familiar objects that are similar, the program asks the following question:

How can you construct a figure with the same shape as another?

One way is to move the figure to various positions by translating it, rotating it, or flipping it over. This gives figures that are *congruent* because they have not only the same shape but also the same size. To change size without changing shape, the program starts with a triangle and introduces *scaling,* the process of multiplying the lengths of all sides of the triangle by the same number, called the *scaling factor*. Two triangles related in this way are said to be *similar*. The lengths of all sides are multiplied by the same factor, but the angles remain unchanged.

Scaling multiplies lengths of *all* line segments by the same factor. Therefore, scaling preserves ratios of lengths of corresponding sides, of corresponding altitudes, of corresponding angle bisectors, and of corresponding medians. Applications show how Thales might have used similar triangles to find the height of a column and of a pyramid by comparing lengths of shadows.

Similarity is also discussed for more general polygons and also for three-dimensional objects. The program then shows what happens to perimeters, areas, and volumes under scaling. Perimeters are multiplied by the scaling factor, areas are multiplied by the square of the scaling factor, and volumes are multiplied by the cube of the scaling factor. The concepts are illustrated with a variety of examples from real life.

Scaling is the basis of all measurement. It plays an important role, not only in geometry and trigonometry, but also in art, science and technology. It reveals the secret of map making, scale drawings and blueprints, and also explains some aspects of photographic images and vision itself. Scaling also explains why a hummingbird's heart beats so much faster than a human heart, and why it is impossible for a small creature such as a praying mantis to become as large as a horse.

II. BEFORE WATCHING THE VIDEOTAPE

Because ratios of numbers are used repeatedly in this program, they are introduced in a section entitled *Review of Prerequisites*. If students are familiar with ratios, this section will serve as a review. If not, an effort should be made to acquaint them with ratios before viewing the rest of the tape. A good way to do this is to have the students read the *Review of Prerequisites* section and solve some of the exercises on pages 7 and 8.

KEY WORDS AND STATEMENTS:

Ratio of two numbers.

The quotient of two numbers, a divided by b, is called the *ratio* of a to b, provided that b is not zero.

THE MAIN IDEAS IN THIS PROGRAM:

Scaling a figure means that all distances in the figure are multiplied by the same positive number, called the *scaling factor*.

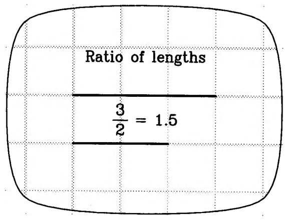
Scaling transforms a figure into a *similar* figure, that is, a figure of the same shape but possibly of different size.

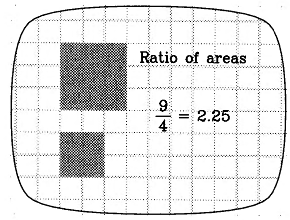
Lengths of corresponding sides of similar polygons have the same ratio, called the *similarity ratio*.

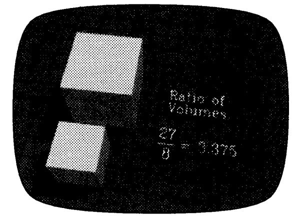
Corresponding angles of similar polygons are equal.

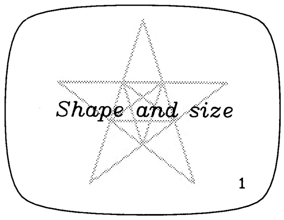
If a figure is scaled by a factor s, lengths of line segments and perimeters are mutliplied by s, surface areas are multiplied by s^2, and volumes are multiplied by s^3.

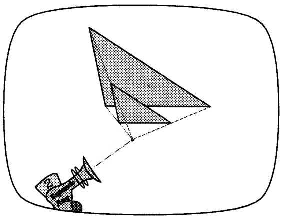
III. TABLE OF CONTENTS

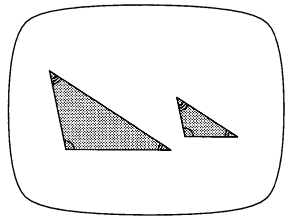


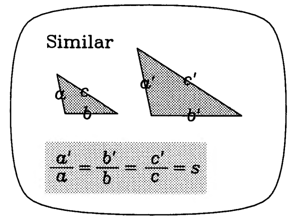


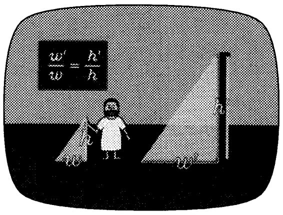


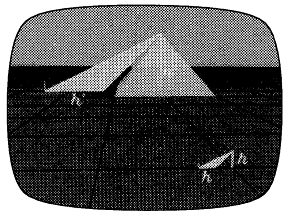


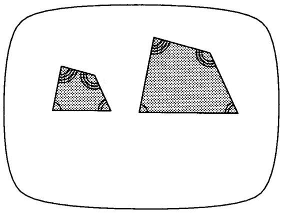


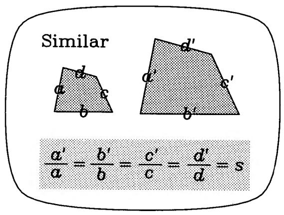


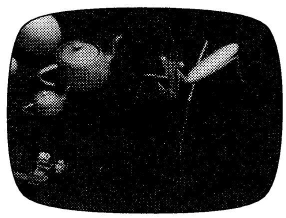


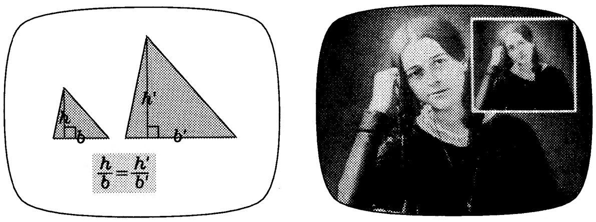


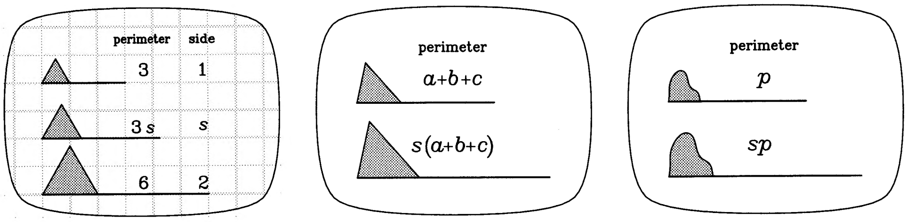


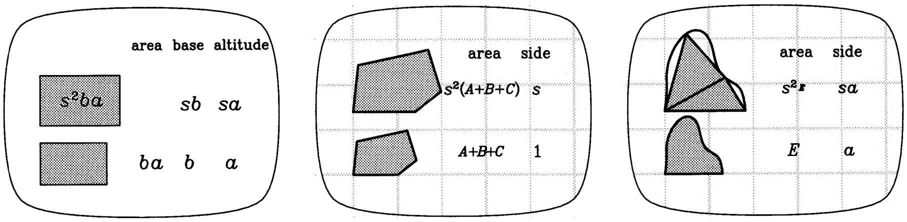


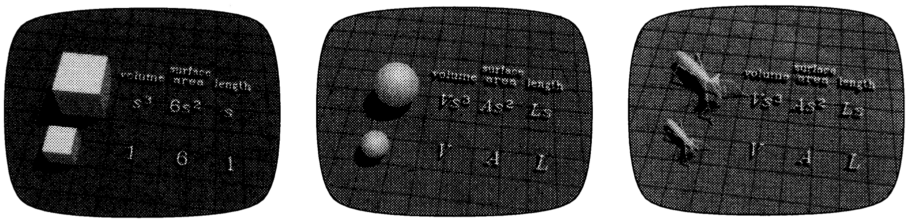





$\frac{h}{b} = \frac{h'}{b'}$


perimeter side
3 1
3s s
6 2
perimeter
a+b+c
s(a+b+c)
perimeter
p
sp


area base altitude
s^2ba sb sa
ba b a
area side
$s^2(A+B+C)$ s
A+B+C 1
area side
sa
E a


volume surface area length
s^3 $6s^2$ s
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volume surface area length
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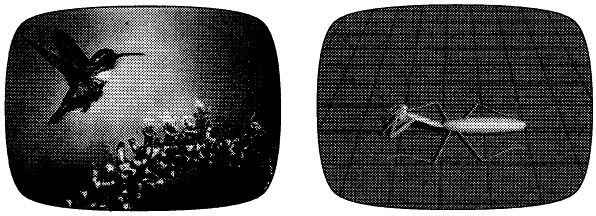
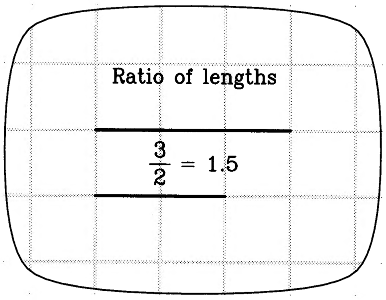
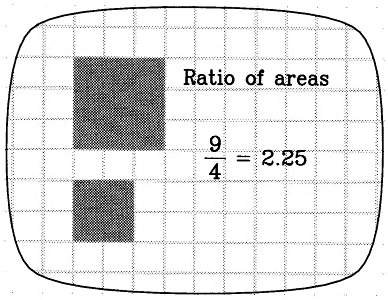
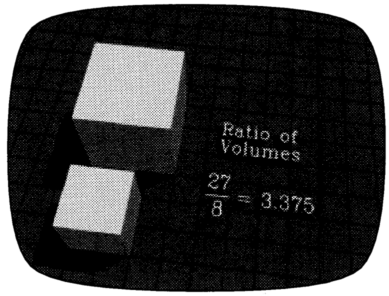
V A L
volume surface area length
Vs^3 As^2 Ls
V A L



Review of prerequisites







This program uses ratios of numbers. *Ratio* is a convenient word used to express a simple idea: one number divided by another. We say the ratio of 8 to 4 is 2 because 8 divided by 4 equals 2:

$$\frac{8}{4} = 2, \text{ the } \textit{ratio} \text{ of 8 to 4.}$$

The ratio of 1 to 4 is 1/4, or 0.25. The ratio of 100 to 400 is also 0.25 because 100 divided by 400 is equal to 0.25. In general, for any two numbers a and b, the number obtained by dividing a by b is called the *ratio* of a to b:

$$\frac{a}{b} = \text{the } \textit{ratio} \text{ of } a \text{ to } b.$$

Of course, the number b in the denominator cannot be zero, because division by zero is undefined.

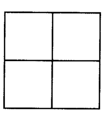
In this program we use ratios to compare various quantities, such as lengths of line segments, areas of plane regions, or volumes of solids. For example, here are two line segments, one of length 2 and one of length 3.

ratio of lengths (shorter to longer) = $\frac{2}{3}$

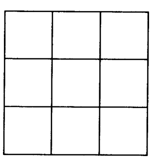
length = 2   length = 3

ratio of lengths (longer to shorter) = $\frac{3}{2}$

These segments can be used to build two squares, one with edge 2 and one with edge 3. The area of the smaller square is equal to 4, the area of the larger is equal to 9.



area = 4



area = 9

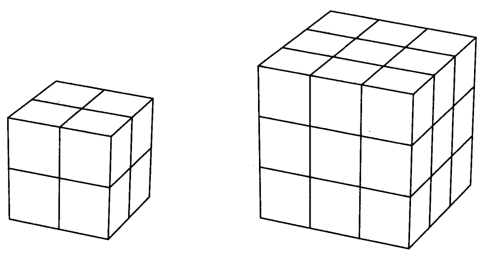
ratio of areas (smaller to larger) = $\frac{4}{9}$

ratio of areas (larger to smaller) = $\frac{9}{4}$

Note that the ratio of the areas is equal to the square of the ratio of the lengths:

$$\frac{4}{9} = (\frac{2}{3})^2 \text{ and } \frac{9}{4} = (\frac{3}{2})^2 .$$

We can also compare volumes of two cubes, one of edge 2 and one of edge 3. The volume of the smaller cube is 8, and the volume of the larger cube is 27.



volume = 8 volume = 27

ratio of volumes (smaller to larger) is $\frac{8}{27} = (\frac{2}{3})^3$

ratio of volumes (larger to smaller) is $\frac{27}{8} = (\frac{3}{2})^3$

Note that the ratio of volumes is equal to the cube of the ratio of lengths.

Algebraic properties of ratios

If r is the ratio of a to b, then the quotient of a divided by b is equal to r, written as follows:

(1) $$\frac{a}{b} = r.$$

This can also be expressed without using the operation of division. Multiplying both sides of Equation (1) by b we find

(2) $$a = br.$$

In other words, if the ratio of one number to another is r, the first number is r times the second. The two equations (1) and (2) are different ways of expressing the same idea.

We have already seen that different pairs of numbers can have the same ratio. For example, the ratio of 2 to 8 is the same as the ratio of 6 to 24 because

$$\frac{2}{8} = 0.25 = \frac{6}{24}.$$

The numerator and denominator in the fraction 6/24 are simply 3 times those in the fraction 2/8. More generally, multiplying the numerator and denominator of a fraction by a nonzero factor s does not change their ratio. In other words,

$$\frac{a}{b} = \frac{sa}{sb} \quad \text{if } s \neq 0.$$

Further properties of ratios are described in the following exercises.

EXERCISES ON RATIOS

In Exercises 1 through 5, you are given two ratios that are equal, say

$$\frac{a}{b} = \frac{c}{d}, \tag{3}$$

where $b \neq 0$ and $d \neq 0$, and you are asked to deduce new information from (3).

1. Show that $\frac{a}{c} = \frac{b}{d}$ if $c \neq 0$.

2. (a) Add 1 to each member of (3) and deduce that $\frac{a+b}{b} = \frac{c+d}{d}$.

   (b) Show that $\frac{a+c}{c} = \frac{b+d}{d}$ if $c \neq 0$.

3. (a) Subtract 1 from each member of (3) and deduce that $\frac{a-b}{b} = \frac{c-d}{d}$.

   (b) Show that $\frac{a-c}{c} = \frac{b-d}{d}$ if $c \neq 0$.

4. Let r denote the common value of the ratios in (3), so that

$$a = br \quad \text{and} \quad c = dr .$$

   (a) By adding these equations, deduce that $\frac{a+c}{b+d} = \frac{a}{b}$, assuming that $b + d \neq 0$.

   (b) If $b - d \neq 0$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

   (c) More generally, if $b + td \neq 0$, show that $\frac{a+tc}{b+td} = \frac{a}{b}$.

   (d) Show that the ratio $\frac{a+tb}{c+td}$ is independent of t if $c + td \neq 0$.

5. (a) If $b^2 + td^2 \neq 0$, show that $\frac{a^2 + tc^2}{b^2 + td^2} = \frac{a^2}{b^2}$.

   (b) State and prove a generalization in which the squares are replaced by higher powers.

6. If $a/x = x/b$, then x is called the *mean proportional* of a and b. If a and b are positive, show that they have exactly two mean proportionals,

$$x = \sqrt{ab} \quad \text{and} \quad x = -\sqrt{ab} .$$

7. Find all values of x for which the ratio of $8 + x$ to $5 + x$ equals (a) 4/5. (b) 8/5.

8. Express the number 77 as a sum of two numbers whose ratio is 5/6.

9. Find all pairs of numbers whose product is 45 and whose ratio is 1/5.

10. Find all numbers x such that the ratio of 1 to x is equal to $x - 1$. (There are two solutions.)

Inequalities involving ratios

11. If a and b are positive, with b greater than a (written $b > a$), show that the ratio of a to b is increased if the same positive quantity x is added to both numerator and denominator. That is, if $b > a$ and $x > 0$, then

$$\frac{a + x}{b + x} > \frac{a}{b}.$$

State and prove a corresponding result when a is greater than b.

12. If a/b and c/d are two unequal ratios of positive numbers, show that the ratio $\frac{a + c}{b + d}$ lies between the two fractions a/b and c/d.

1. *Shape and size*

In geometry, similar objects are those that have the same shape but not necessarily the same size. Before we discuss the mathematical definition of similarity we illustrate the concept with a number of examples. Figure 1 shows examples of similar figures in the plane.

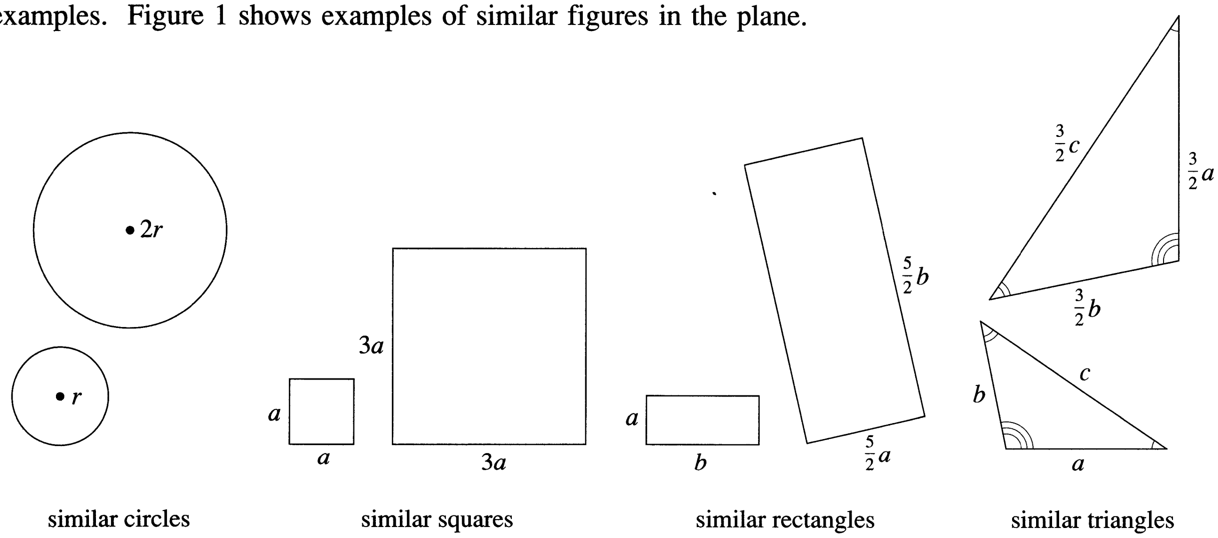


Figure 1. Examples of similar figures in the plane.

In Figure 1, the radius of the larger circle is twice that of the smaller circle. The edges of the large square are three times as long as those of the small square. The edges of the large rectangle are 5/2 times as long as those of the small rectangle. And the large triangle has edges 3/2 times as long as those of the small triangle. In each case, the larger figure is a magnification of the smaller figure by a scaling factor. The scaling factor is 2 for the circles, 3 for the squares, 5/2 for the rectangles, and 3/2 for the triangles.

In all these examples of similar figures, each is a scale copy of the other. For polygons, this means that corresponding angles are equal and all the corresponding lengths are multiplied by a constant scaling factor. Similar figures are said to have the same shape. If the constant scaling factor is 1 the figures are also called *congruent*.

Any two circles are similar, any two squares are similar, and any two equilateral triangles are similar. Figure 2 shows the letter F and several similar copies. Note that if any one of these is moved without changing the lengths of its sides, for example, if it is shifted, rotated, or flipped over to form a mirror image, the congruent copy obtained is similar to the original F.

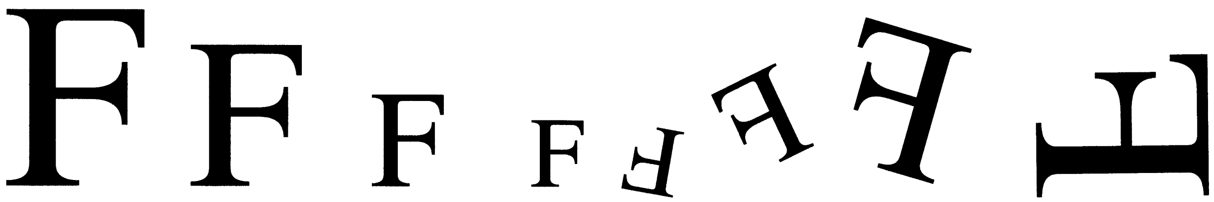


Figure 2. Similar copies of the letter F.

Two arbitrary rectangles are not necessarily similar, even though all their angles are equal. The next diagram shows examples of rectangles and other figures that are *not* similar because corresponding sides are not multiplied by the same scaling factor.

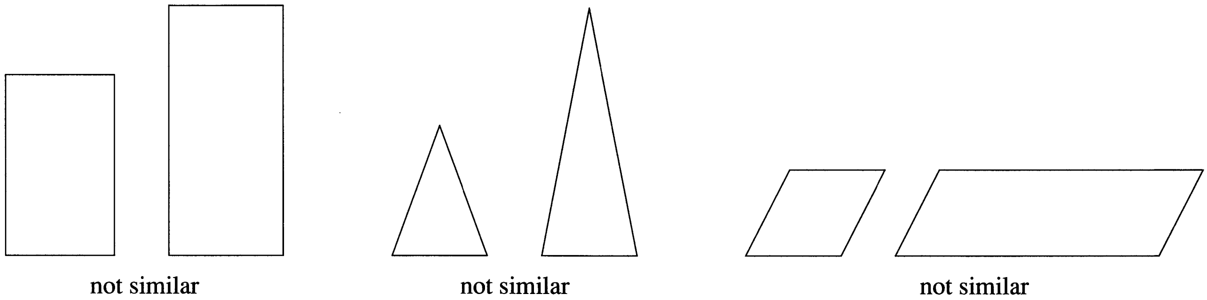


Figure 3. Examples of plane figures that are not similar.

The three quadrilaterals in Figure 4 have edges of equal length. Nevertheless, they are not similar because corresponding angles are not equal.

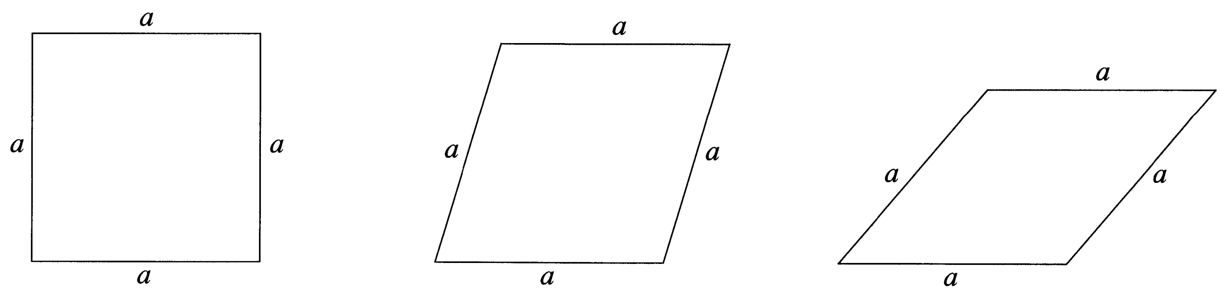


Figure 4. Plane figures that are not similar because corresponding angles are not equal.

Similarity applies not only to plane figures, but to solid objects as well.

***Mathematical definition of scaling*.** Choose a fixed point O, which we call the origin, and a fixed positive number s, which we call the scaling factor. For every point P we associate another point P' whose distance from O is equal to s times the distance of P from O. The process of associating P' with P is called *scaling by a factor s*, and the point O is called the *center of scaling*.

We denote the point P' as sP. As point P traces out a figure, the corresponding point sP traces out a scaled copy of this figure, as shown by the example in Figure 5. If P traces out a triangle, then sP traces out a similar triangle. The points can lie in a plane or in 3-dimensional space. For example, if P traces out a sphere of radius r, the point sP traces out a sphere of radius sr.

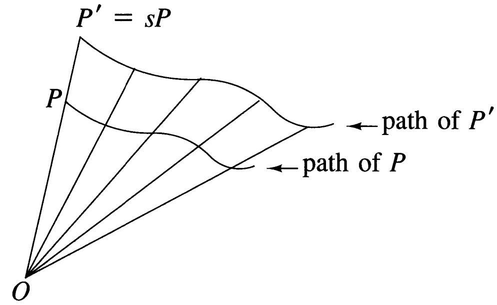


Figure 5. Scaling multiplies all distances from O by the scaling factor.

Figure 6 illustrates a fundamental property of scaling. Two points P and Q at distance d apart are shown with the associated points sP and sQ, whose distances from O are s times those of P and Q from O. It can be shown that the line segment joining P and Q is parallel to the segment joining sP and sQ, and that the distance from sP to sQ is equal to s times the distance d from P to Q. Moreover, this property is independent of the choice of origin O. In other words, scaling by a factor s multiplies distances between *any* two points by s, regardless of the location of the center of scaling. Also, the measure of any angle is preserved by scaling.

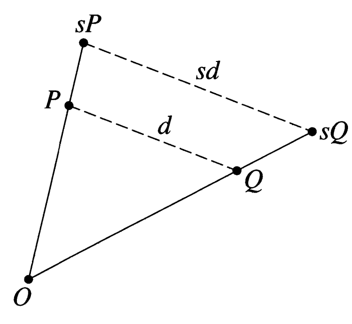


Figure 6. Scaling by a factor s multiplies all distances by the factor s.

This program focuses on properties shared by scaled figures. If you know something about one shape, the theory of similarity tells you something about every scaled copy. Corresponding angles are equal, and lengths of corresponding line segments have the same ratio. We'll also learn what happens to areas of plane figures or volumes of solids when figures are scaled.

EXERCISES

In each of Exercises 1 through 6, determine which figures are similar. For each pair of similar figures, use a ruler to determine the scaling factor that converts the smaller figure to the larger, and also determine the corresponding sides.

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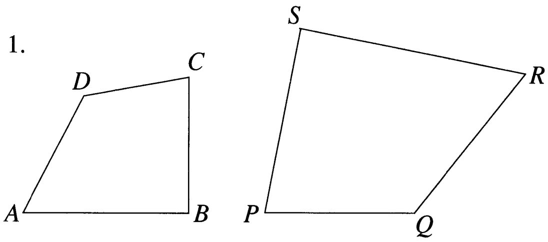

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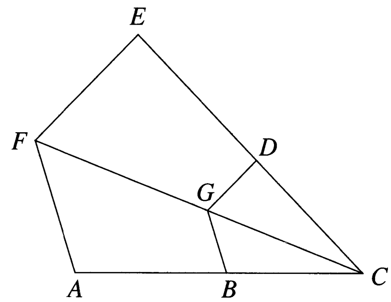

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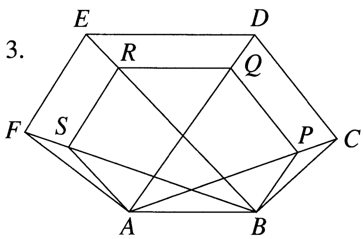

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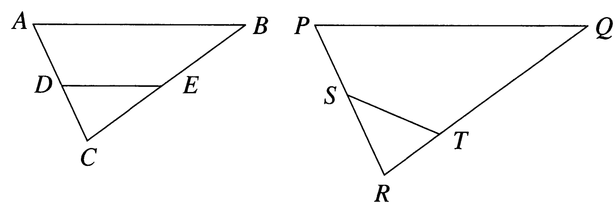

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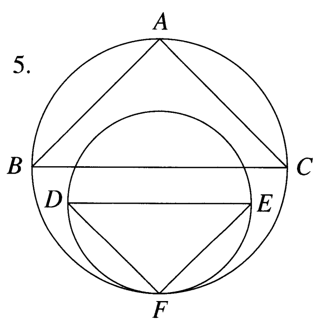

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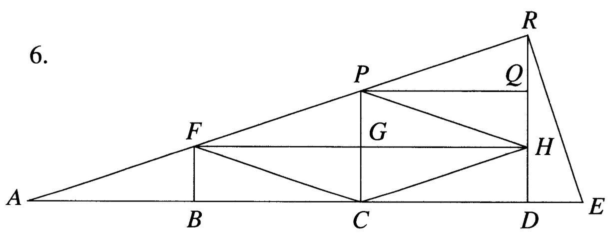


7. (a) Given any triangular shaped tile. Show that four copies can be fitted together to form a larger triangle similar to the given triangle.

(b) Is there a triangular tile such that *three* copies fitted together form a larger similar triangle?

(c) Is there a triangular tile such that *two* copies fitted together form a larger similar triangle?

8. (a) A 20 ft pole and a 30 ft pole, each perpendicular to the ground, are a certain distance apart. Two cables (straight line segments) are used to connect the top of each pole to the bottom of the other. Show that the cables intersect at a point whose height above the ground does not depend on the distance between the poles.

(b) If the poles in part (a) have lengths a and b, determine the height of the intersection of the cables in terms of a and b.

9. Given two intersecting lines and a point P not on either line. Use similarity to show that there is a circle that passes through P and is tangent to each of the lines. How many such circles are there?

2. *Similar triangles*

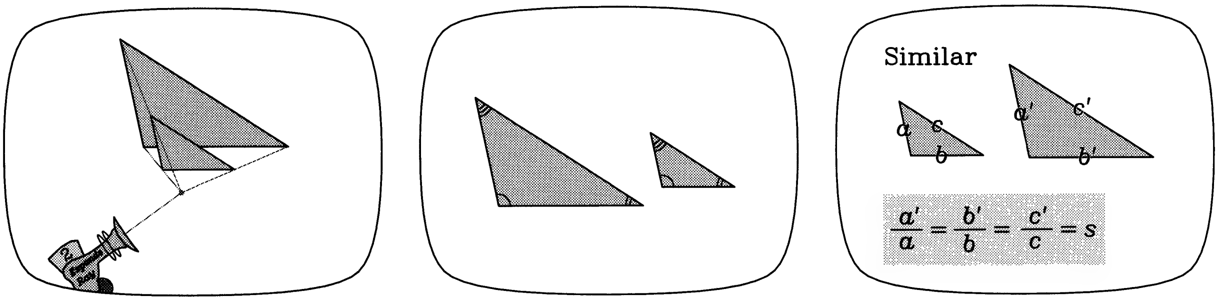


Figure 7 shows two similar triangles. They have the same shape because corresponding angles are equal, but are of different size, the edges of the larger triangle being twice as long as those of the smaller. We label the lengths of the edges of the smaller triangle a, b, c and the corresponding lengths in the larger triangle a', b', c'. In Figure 7, b and b' are the shortest lengths, c and c' are the longest. Because the sides of the larger triangle are twice as long as those of the smaller, we have $a' = 2a$, $b' = 2b$ and $c' = 2c$, from which we find that lengths of corresponding sides have the same ratio:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = 2.$$

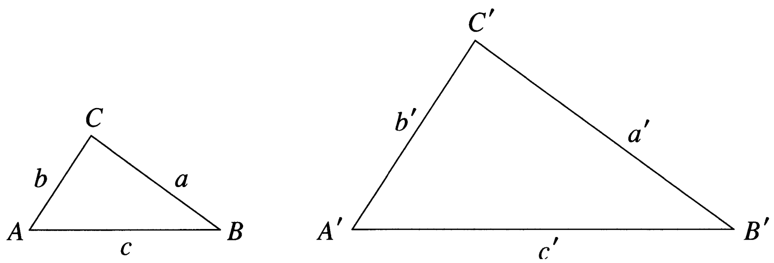


Figure 7. Similar triangles.

For arbitrary similar triangles, the common ratio of lengths of corresponding sides is the scaling factor, also called the *similarity ratio*. The scaling factor s is called an *expansion factor* if s is greater than 1, and a *contraction factor* if s is less than 1. If the scaling factor is equal to 1, the triangles are congruent.

In Figure 7 the sides of the larger triangle are twice as long as those of the smaller triangle, and we say that the larger triangle is obtained from the smaller by expansion by the factor $s = 2$. On the other hand, the smaller triangle is obtained from the larger by contraction by the factor $s = 1/2$.

If triangles ABC and $A'B'C'$ are similar, corresponding angles are equal (have the same measure),

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C'.$$

In other words, expansion or contraction of a triangle by a scaling factor does not change the angles. Conversely, if corresponding angles of two triangles are equal, it can be shown that lengths of corresponding sides have the same ratio, so the triangles are similar. We say they have the same shape because corresponding angles are equal, but the triangles can be of different size.

If the sides of a triangle are scaled by a factor, then the lengths of *all* line segments are scaled by the same factor. For example, to see why the altitudes are scaled by the same factor, refer to Figure 8 which shows two similar triangles with similarity ratio s, so that $a' = sa$, $b' = sb$, $c' = sc$. An altitude CP of length h is drawn from vertex C perpendicular to the opposite side AB. Because scaling preserves angles, the corresponding side $C'P'$ of triangle $A'B'C'$ is perpendicular to $A'B'$. Let h' denote the length of $C'P'$. The altitude CP divides triangle ABC into two right triangles CPB and CPA. The two right triangles CPB and $C'P'B'$ are similar because corresponding angles are equal, and hence lengths of corresponding sides have the same ratio: $h'/h = a'/a$. But $a'/a = s$, so $h'/h = s$, and hence $h' = sh$. This shows that lengths of corresponding altitudes are scaled by the same factor.

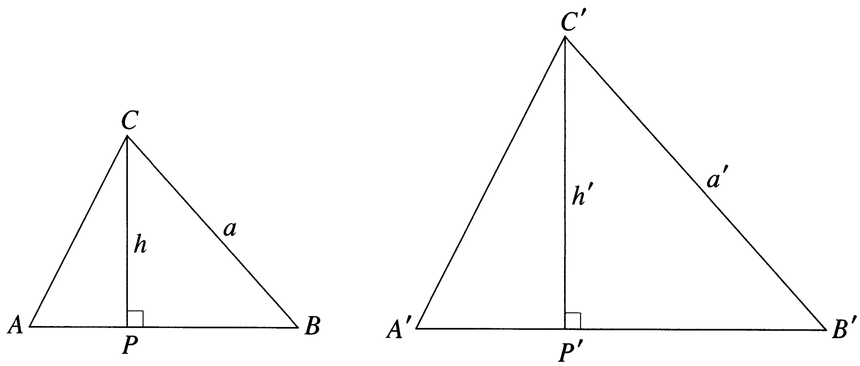


Figure 8. Corresponding altitudes of similar triangles are scaled by the same factor.

The same argument applies to any segments CQ and $C'Q'$ drawn so that $\angle CQA = \angle C'Q'A'$, as shown in Figure 9. For example, we could take CQ to be the bisector of angle ACB, or a median, the segment from vertex C to the midpoint of the opposite side. In any case, if $\angle CQA = \angle C'Q'A'$ the two triangles CQA and $C'Q'A'$ are similar because their corresponding angles are equal, and when we equate ratios of lengths of corresponding sides we find that the length of $C'Q'$ is equal to s times the length of CQ. These illustrate the fact that scaling a triangle changes lengths of all line segments by the same factor.

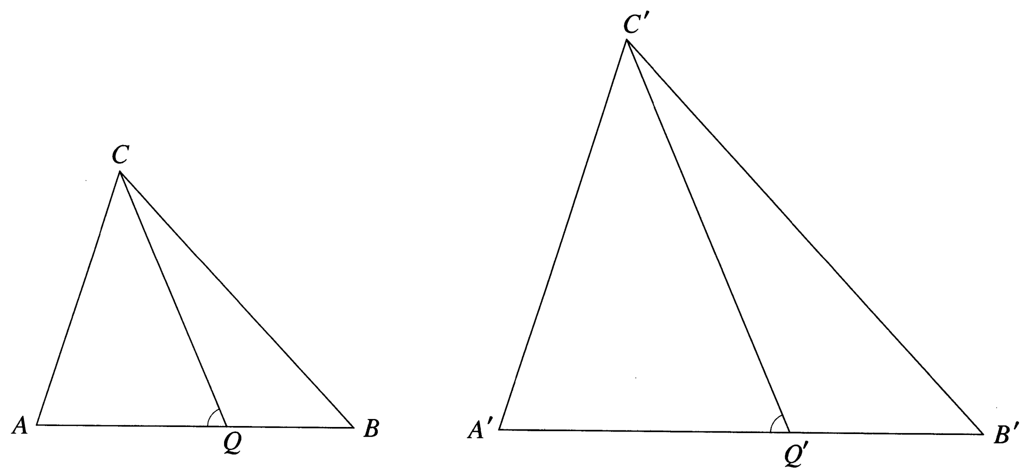


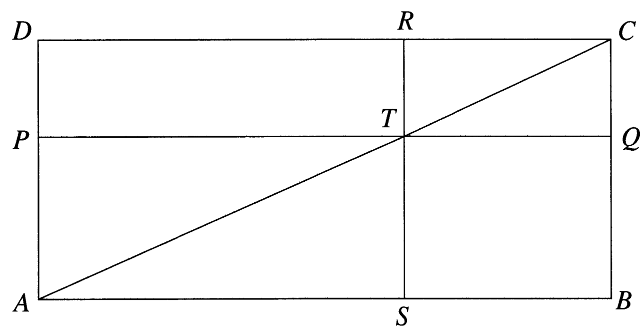
Figure 9. Corresponding segments in similar triangles are scaled by the same factor.

EXERCISES ON SIMILAR TRIANGLES

1. The edges of a triangle have lengths 3, 5 and 7. Find the lengths of the edges of a similar triangle whose perimeter is 42.

2. If two triangles ABC and $A'B'C'$ have two of their corresponding angles equal, say $\angle A = \angle A'$ and $\angle B = \angle B'$, show that the third angles are also equal: $\angle C = \angle C'$.

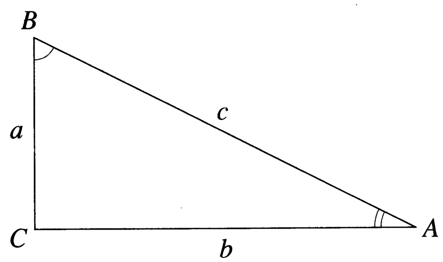
3. In the following figure, $ABCD$ is a rectangle, PQ is parallel to AB and intersects RS, which is parallel to DA, at a point T on the diagonal AC. Prove that the three triangles AST, TQC, and ABC are similar.

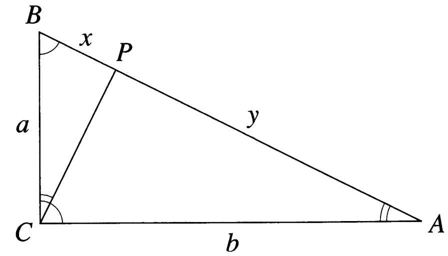


4. The altitude from the right angle to the hypotenuse of a right triangle divides the hypotenuse into segments of length 4 and 6. How long is the altitude? How long is it if each segment has length 6?

5. The altitude from the right angle to the hypotenuse of a right triangle divides the hypotenuse into segments of unequal length. Find the ratio of the longer segment to the shorter, given that one leg of the right triangle is three times as long as the other.

6. This exercise outlines a proof of the Theorem of Pythagoras based on similar triangles. Right triangle ABC has legs of length a and b, and hypotenuse of length c. Line CP is constructed perpendicular to the hypotenuse AB, forming two new right triangles APC and BPC, and dividing the hypotenuse into segments of length x and y, as shown.





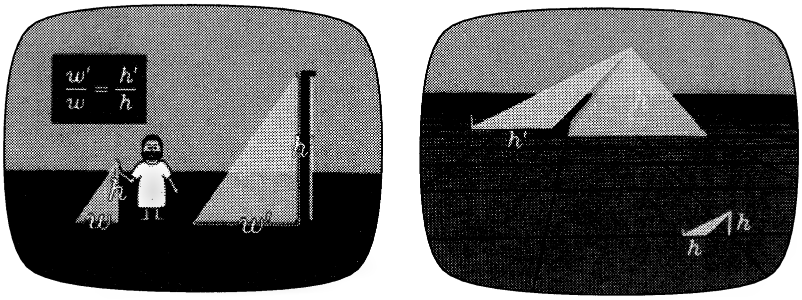
(a) Prove that triangles ABC and CBP are similar, and that $x/a = a/c$.

(b) Prove that triangles ABC and ACP are similar, and that $y/b = b/c$.

(c) Part (a) shows that $a^2 = cx$, whereas part (b) shows that $b^2 = cy$. Add these equations to deduce the Theorem of Pythagoras:

$$a^2 + b^2 = c^2.$$

3. *Applications of similarity*



One of the earliest applications of similarity was to determine the height of a tall object, such as a tree or column, without measuring its length directly. The Greek mathematician Thales in the 6th century B.C. is said to have invented a method for determining the height of a column by comparing the length of its shadow with that of his staff. Figure 11 shows a column whose height h' is to be determined.

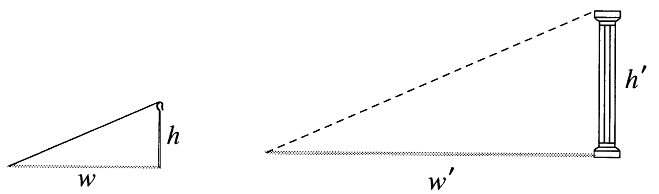


Figure 11. The method of Thales for determining the height of a column.

The quantities that are known or can be measured are the height of Thales's staff, h, and the lengths of the two shadows, w and w'. A right triangle is formed by drawing the dotted line from the end of the shadow of Thales's staff to the top of the staff. Another right triangle is formed by drawing the dotted line from the end of the shadow of the column to the top of the column. If the shadows are cast at the same time of day, the dotted lines, which represent the direction of the sun's rays, can be regarded as parallel because the sun is so far away. Therefore the two triangles are similar because corresponding angles are equal. Consequently, corresponding sides have the same ratio: $h'/h = w'/w$. Solving this equation for h' we find

$$h' = h\,(\frac{w'}{w}).$$

In other words, the height of the column, h', is equal to the height of the staff, h, multiplied by the ratio of the lengths of the shadows.

This simple solution can be simplified even further if the shadows are measured at the time of day when the shadow length w is exactly equal to the height h of the staff. When $w = h$, the equation $h' = hw'/w$ simplifies to $h' = w'$. It seems likely that Thales would have used this simplified method because it avoids two arithmetical calculations: determining the ratio of the shadow lengths, and multiplying this ratio by h. Although these calculations appear trivial in the modern world of hand-held calculators, they were not trivial in ancient times. The introduction of Arabic numerals, decimal notation, and their use in doing simple arithmetical operations such as multiplication and division, came into being many centuries after the time of Thales.

Legend has it that Thales also calculated the height of a pyramid by comparing lengths of shadows. This is more of a challenge because part of the pyramid shadow falls on the pyramid itself. Figure 12 shows a simple way to solve the problem.

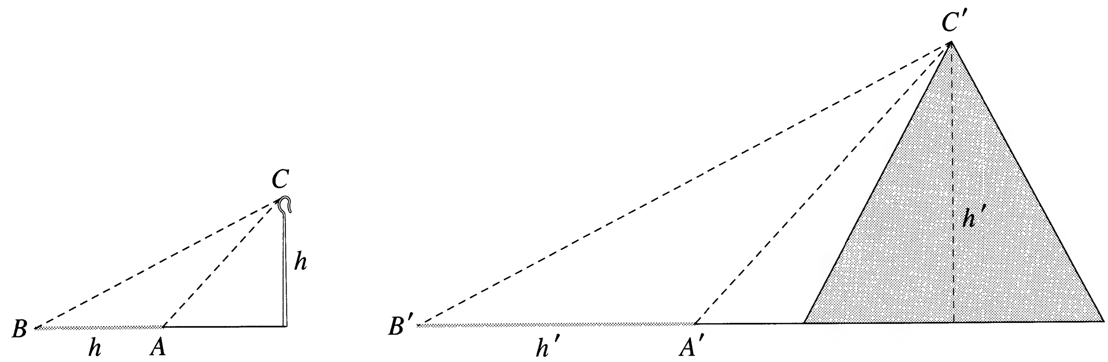


Figure 12. Determining the height of a pyramid by measuring changes in shadow lengths.

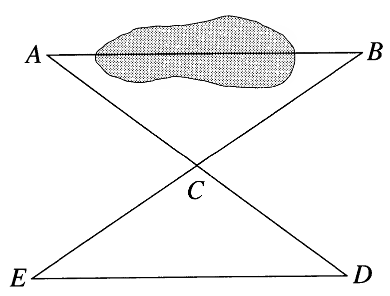
At a convenient time of day when the end of the pyramid shadow falls outside the pyramid, place a marker at A' and a corresponding marker at A, marking the end of the staff shadow. At a later time of day when the length of the staff's shadow is stretched by an amount equal to the height of the staff, place a marker at point B marking the end of the staff's shadow and a corresponding marker at point B' marking the end of the pyramid's shadow. The two triangles ABC and $A'B'C'$ are similar because corresponding angles are equal. The altitude of triangle ABC has length h, the height of the staff, and this is equal to the length of side AB. Consequently the height h' of the pyramid, which is the length of the altitude of triangle $A'B'C'$, is equal to that of side $A'B'$. Therefore the pyramid height h' is equal to the change in length of the pyramid's shadow. Note that the altitude of the pyramid need not lie in the plane of triangle $A'B'C'$.

EXERCISES USING SIMILAR TRIANGLES TO CALCULATE LENGTHS

The following exercises use similar triangles to determine the length of a line segment that cannot be measured directly because all or part of the segment is inaccessible.

1. In this exercise, A and B are separated by a pond, and we wish to determine the distance AB. In this situation, both points A and B are accessible from a third point C not on the line AB.

(a) Extend the line segment through AC to the point D such that $CD = AC$ (in length). Extend BC to the point E such that $CE = BC$, as shown in the figure. Use congruent triangles to show that $AB = ED$.



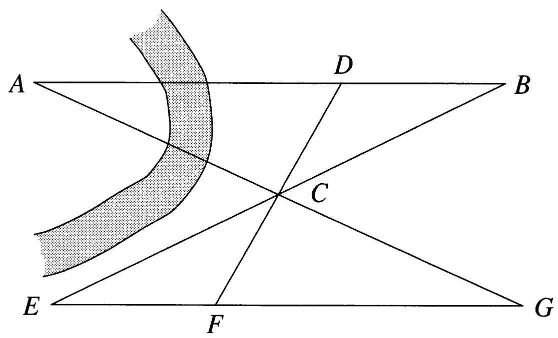
Exercise 1. Endpoints A and B accessible from C.

(b) Refer to part (a), but extend the line segment through AC until CD is half the length AC, and extend the segment through BC until CE is half the length BC. Use similar triangles to show that $AB = 2ED$.

2. In this case, we wish to determine distance AB when A and B are separated by a river, with both A and B visible from a third point C not on AB, but with only one endpoint B accessible from C.

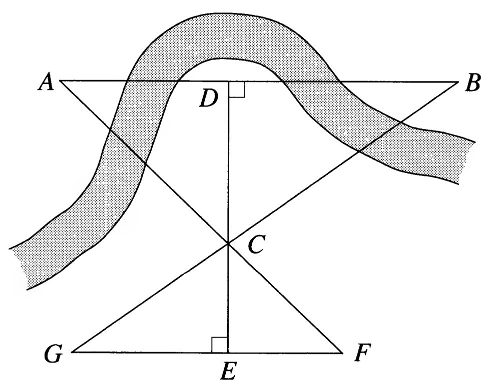
(a) Extend the line segment through BC until $CE = BC$. On the line through A and B, choose a convenient point D accessible from C and extend the segment through DC until $CF = DC$. Let G be the point of intersection of the lines through AC and EF, as shown in the figure. Show that $AB = EG$.

(b) Refer to part (a), but extend the line segment through BC until CE is one-third the length BC, and extend the segment through DC until CF is one-third DC. Show that $AB = 3EG$.



Exercise 2. Only one endpoint B is accessible from C.

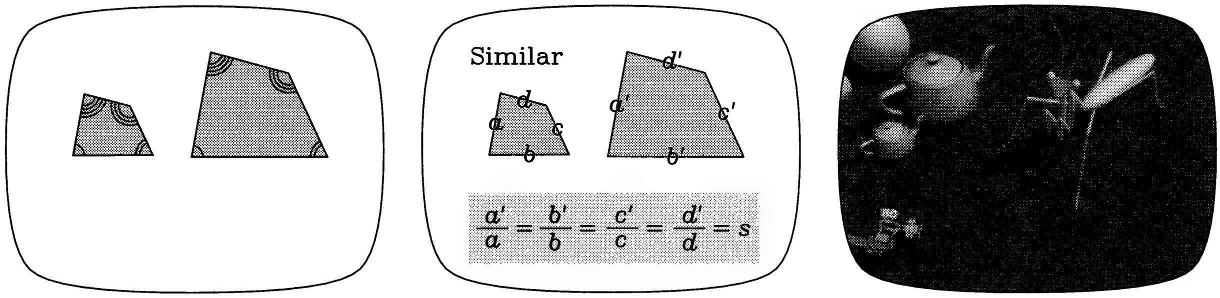
3. In this example, both A and B are visible from a third point C not on the line AB, neither is accessible from C, but a line through C perpendicular to AB intersects AB at a point D accessible from C. Extend the line segment through DC until $CE = sDC$, where s is a convenient scaling factor. Draw a line through E perpendicular to CE. On this line, let F be the point collinear with A and C, and let G be the point collinear with B and C. Show that $AB = FG/s$.



Exercise 3. Both endpoints A, B are visible but not accessible from C.

4. In this example, the entire line segment AB is inaccessible, but the segment can be extended beyond A and B to two points D and E that are accessible from a third point C not on the line through AB. Explain how to determine the length of the segment AB.

4. *Similar polygons and solids*



The definition of scaling described in Section 1 applies equally well to polygons and other plane figures, and to solid objects. If we start with a figure F, such as the quadrilateral shown in Figure 13, we can make a scaled copy of F and then move the scaled copy around by translating it, rotating it, or reflecting it through a line. Any figure obtained by performing one or more of these operations will be similar to F.

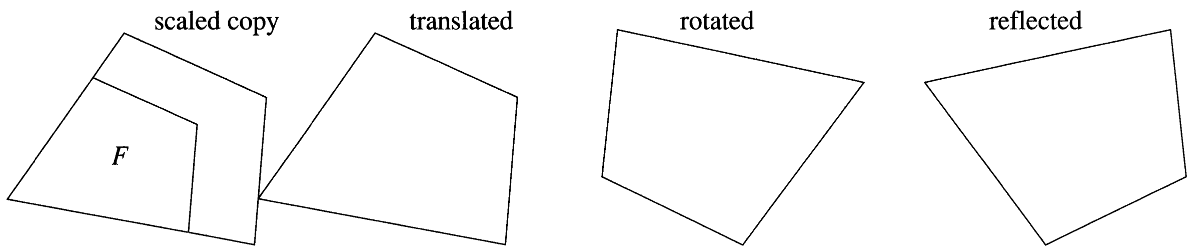


Figure 13. Similar figures related by scaling, translation, rotation, and reflection.

When similar shapes are moved around as in Figure 13, it may take a moment to figure out the corresponding sides. It often helps to match the shortest sides, and the longest sides. Also, try to match angles that appear to be equal in size.

In three-dimensional space, expanding or contracting a given solid by a scaling factor s always produces a similar solid. As in the plane, when all distances from a fixed center of scaling are multiplied by a factor s, the distance between any two points is also multiplied by s, regardless of the location of the center of scaling. Scaling, translating, rotating, or reflecting a solid object always gives a similar object. Any two spheres are similar, and any two cubes are similar. Figure 14 shows examples of similar solids, and Figure 15 shows examples of solids that are not similar.

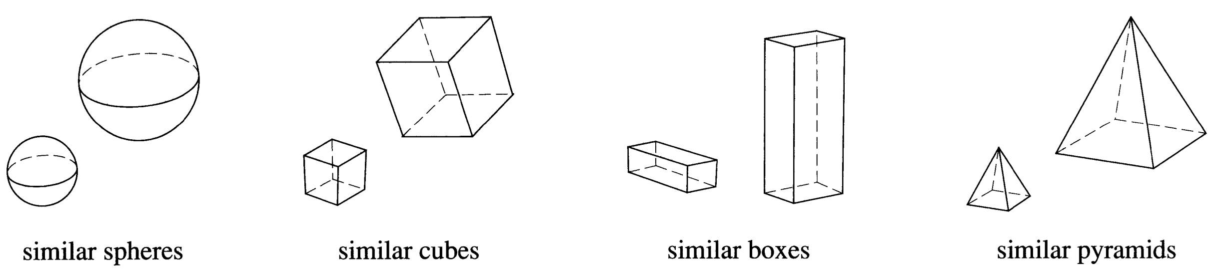


Figure 14. Examples of similar solids.

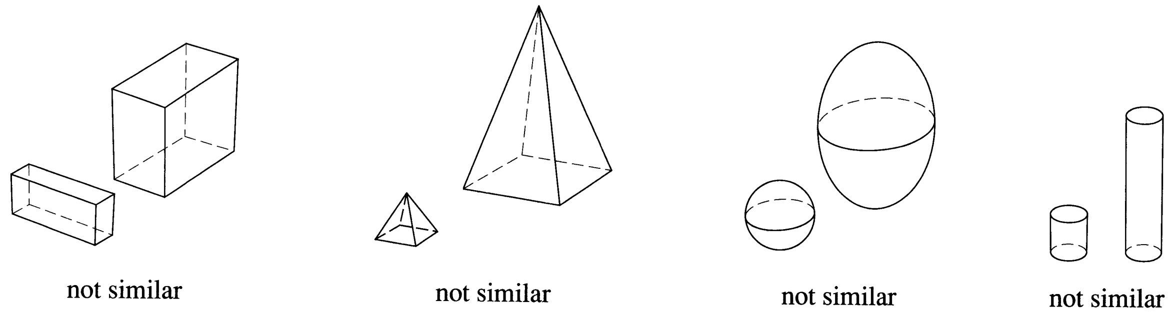


Figure 15. Examples of solids that are not similar.

Engineers who design manufactured objects often build scale models to check design features that depend primarily on shape and not on size. Scale models of automobiles or aircraft can be tested in wind tunnels to check air flow and measure wind resistance. It is easier and less expensive to make design changes in scale models before manufacturing full size objects.

5. *Internal ratios of similar figures*

Commerce and science share a common need for standardized units of measure. Throughout history, governments and scientific organizations have tried to establish standard units of time, length, area, and weight. When measuring lengths, a convenient unit of length is agreed on, such as the inch, centimeter, foot, meter, or mile; lengths are measured in terms of this unit. To convert from one unit to another we multiply distances by an appropriate factor. For example, there are twelve inches in one foot, so distances in feet are multiplied by 12 to convert them to inches.

In geometry, lengths are often given without specific mention of the unit length. For example, if a line segment is said to have length 3, it is understood that the segment is 3 times as long as the unit length. Once a choice of unit has been made, two perpendicular line segments having lengths 3 and 4 determine a right triangle with hypotenuse of length 5. The 3-4-5 right triangle in Figure 16 is drawn with the centimeter as unit length:

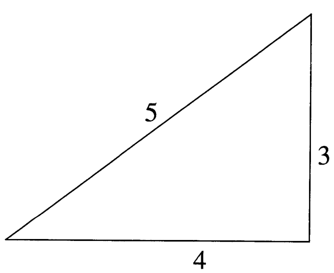


Figure 16.

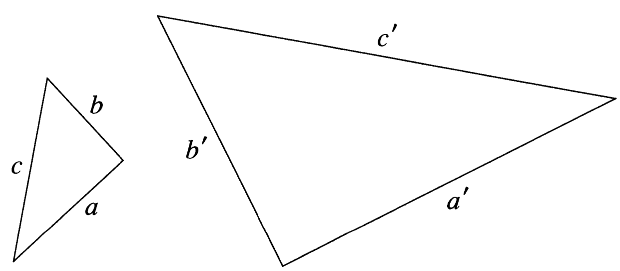


Figure 17.

Figure 17 shows two triangles, each of which is similar to the 3-4-5 right triangle in Figure 16. Because these triangles are similar, lengths of corresponding sides have the same ratios. For example,

$$\frac{a}{4} = \frac{b}{3} \text{ and } \frac{a'}{4} = \frac{b'}{3}.$$

These equations can be rearranged to give $3a = 4b$ and $3a' = 4b'$, or

$$\frac{a}{b} = \frac{4}{3} \quad \text{and} \quad \frac{a'}{b'} = \frac{4}{3}.$$

The two ratios a/b and a'/b' are called *internal ratios* because they compare lengths in the same figure. This example illustrates that internal ratios of similar figures are equal:

$$\frac{a}{b} = \frac{a'}{b'}.$$

In the same way, by equating ratios of corresponding sides

$$\frac{b}{3} = \frac{c}{5} \quad \text{and} \quad \frac{b'}{3} = \frac{c'}{5},$$

we find that the following internal ratios are equal:

$$\frac{b}{c} = \frac{b'}{c'} = \frac{3}{5}.$$

Also, we have

$$\frac{a}{4} = \frac{c}{5} \quad \text{and} \quad \frac{a'}{4} = \frac{c'}{5},$$

from which we find

$$\frac{a}{c} = \frac{a'}{c'} = \frac{4}{5}.$$

More generally, any two scaled figures always have the same internal ratios. To see why, suppose we have two similar figures related by a scaling factor s. Form the ratio of two lengths in one of the figures, say a'/b'. Because the scaling factor is s, the corresponding lengths in the scaled figure are $a' = sa$ and $b' = sb$. When we form the ratio a'/b' the scaling factor s cancels and we find

$$\frac{a'}{b'} = \frac{a}{b}.$$

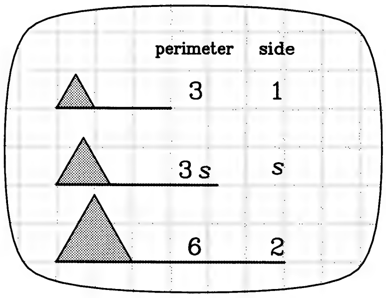
This is true for all internal ratios formed from lengths of sides, altitudes, medians, angle bisectors, and all other line segments. The ratio of two lengths in one figure is equal to the ratio of the corresponding lengths in any scaled copy of the figure.

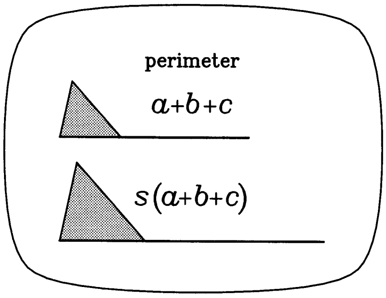
Suggested Project:

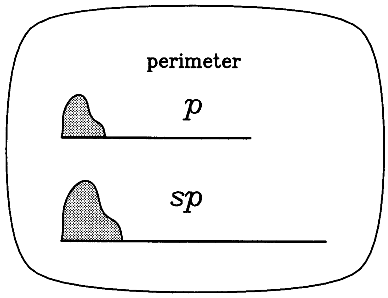
Select a snapshot of a person and an enlargement. Use a ruler to measure two distances in the snapshot and calculate their ratio; for example, the ratio of nose length to distance between the eyes, or the ratio of head length to arm length. Do the same in the enlargement, and verify that these internal ratios are very nearly equal in both copies of the photograph. That's how we recognize the same person in both photographs.

Note. When measurements are made on actual photographs, the calculated values of the ratios may not be exactly equal. How do you account for such discrepancies?

6. *Perimeters of similar figures*







The perimeter of a triangle is the sum of the lengths of its sides, $a + b + c$. If each side is multiplied by a factor s, the perimeter is also multiplied by s because $sa + sb + sc = s(a + b + c)$. This can be stated as follows:

Scaling a triangle by a factor s multiplies its perimeter by s.

The same is true for more general figures. For example, the perimeter of a polygon P is the sum of the lengths of its sides. Scaling by a factor s multiplies the length of each side by s, so the perimeter of the scaled polygon is s times that of P:

Scaling a polygon by a factor s multiplies its perimeter by s.

The same is true for a circle. Scaling by a factor s multiplies the diameter and perimeter of a circle by the same factor s. The perimeter is multiplied by s because it is the limiting value of perimeters of approximating polygons. The perimeter of a circle is also called its *circumference.*

If we start with a circle of diameter 1 and scale by a factor d we obtain another circle with both diameter and circumference multiplied by d. Therefore the internal ratio of circumference to diameter is unchanged by scaling because the scaling factor d cancels when we form the ratio. In other words, the ratio of circumference to diameter is the same for all circles. This ratio, a fundamental constant of nature, is denoted by the Greek letter π, the first letter of the Greek word for *perimeter.* The circumference of any circle of diameter d is equal to πd (or $2\pi r$, in terms of the radius r). In another program, entitled *The Story of* π, we'll learn that π is not a rational number (the ratio of two integers), but it can be approximated to any degree of accuracy by rational numbers. Two common rational approximations are 22/7, which gives π correct to two decimals, and 355/113, which gives six decimals of π.

EXERCISES

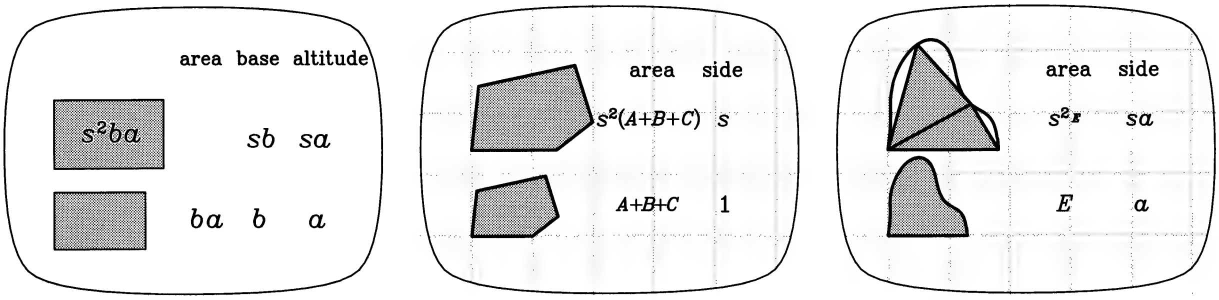
1. Choose several circular objects (for example, jar lids, pots and pans, hula hoops, bicycle wheels) whose circumference C and diameter d can be conveniently measured with a tape measure. For each object, record the measured values of C and d, and calculate the ratio of these measurements, C/d.

(a) Explain why the calculated values C/d may differ slightly for objects of different sizes, even though the theory of similarity says that this ratio is independent of the radius.

(b) If different people measure the same objects, it is likely that their tabulated values for C, d, and C/d will not be identical. What do you think are the reasons for these differences?

2. Pretend that you are in ancient Egypt around 3,000 B.C. and want to estimate the value of π. Your only tools are wooden stakes and ropes. You have no compass, no pencil or paper, no calibrated measuring tape, and no arithmetic as we know it today (with Arabic numerals and decimals). Assume you can find a flat patch of moist sand along the banks of the Nile. Explain how you would draw a circle in the sand, how you would measure its circumference and diameter, and how you would determine a value for the ratio of circumference to diameter.

7. *Areas of similar figures*



If two triangles are similar with similarity ratio s, lengths of corresponding sides have ratio s, and the same is true for the ratios of corresponding altitudes. (See Figure 8 in Section 2.) Because the area of a triangle is one-half base times altitude, if the base and altitude are each multiplied by s, the area is multiplied by s^2. Therefore, if two triangles are similar with similarity ratio s, the ratio of their areas is s^2. This can be stated as follows:

Scaling a triangle by a factor s multiplies its area by s^2.

A general polygonal figure can be decomposed into triangular pieces. An example is shown in Figure 18. If we let A denote the area of the polygonal figure and let A_1, A_2, ..., A_5, denote the areas of the triangular pieces, then A is the sum of the areas of the triangular pieces:

$$A = A_1 + A_2 + A_3 + A_4 + A_5.$$

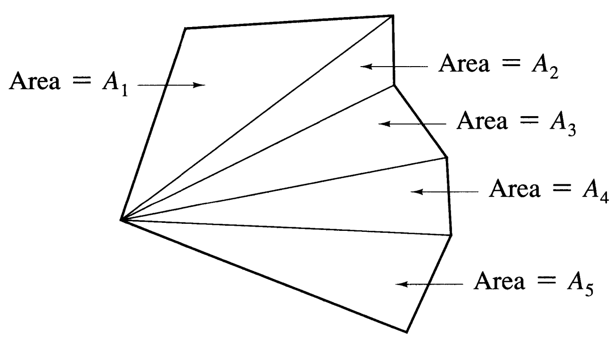


Figure 18. A polygonal figure decomposed into five triangular pieces.

If the polygonal figure is scaled by a factor s, each triangular piece is scaled by the same factor, and the area of each triangular piece gets multiplied by the factor s^2. Thus the scaled polygon has area

$$s^2A_1 + s^2A_2 + s^2A_3 + s^2A_4 + s^2A_5.$$

The common factor s^2 can be factored out and we get

$$\text{area of scaled polygon} = s^2(A_1 + A_2 + A_3 + A_4 + A_5) = s^2A.$$

The same argument works for any polygon.

Scaling a polygon by a factor s multiplies its area by the factor s^2.

Now take a more general plane figure F with curved boundaries, and let A denote its area. Scaling by a factor s produces a similar figure F', as shown by the example in Figure 19.

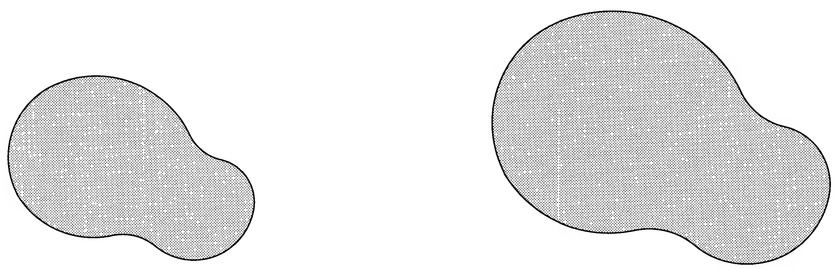
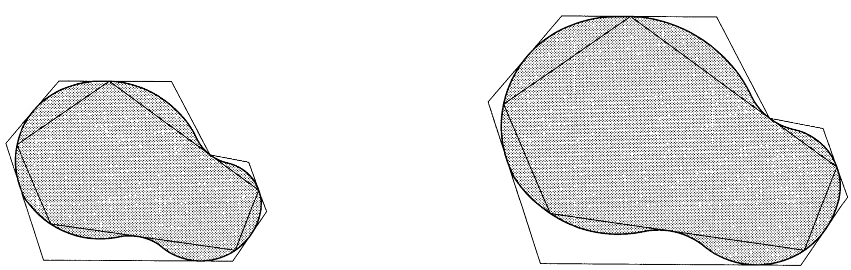


Figure F has area A Scaled figure F' has area s^2A

Figure 19. Scaling a plane figure by a factor s multiplies its area by s^2.

The area of F' is equal to s^2A. This is because F can be approximated from the inside and from the outside by polygons, as shown in Figure 20(a). The area of F lies between the areas of the inner and outer polygons:

$$\text{area of inner polygon} < A < \text{area of outer polygon}.$$



(a) (b)

Figure 20. Curved shapes approximated by inner and outer polygons.

By taking approximating polygons with an increasing number of sides, the areas of the inner and outer polygons can be made arbitrarily close to each other and therefore arbitrarily close to the area of F. If

the entire diagram is scaled by a factor s, the approximating polygons are also scaled by a factor s, illustrated in Figure 20(b), and their areas are multiplied by the factor s^2:

$$s^2(\text{area of inner polygon}) < \text{area of } F' < s^2(\text{area of outer polygon}).$$

Because the areas of the inner and outer polygons can be made arbitrarily close to A, it follows that the area of F' is equal to s^2A. [This is an application of the 'squeezing principle' in the theory of limits. If a fixed number a lies between two changing quantities x and y, and if both x and y approach the same limit z, then $a = z$.] Therefore we have the following property:

Scaling a plane figure by a factor s multiplies its area by s^2.

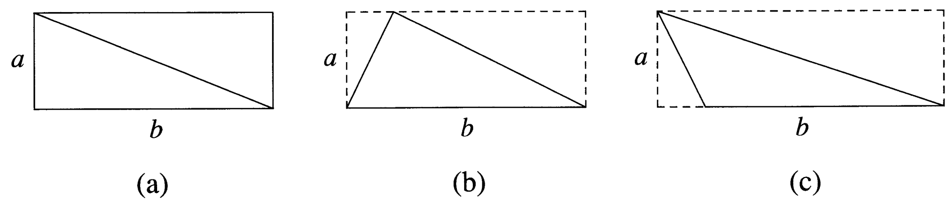
Circumference and area of a circular disk

The foregoing ideas can be applied to circles. Any two circles are similar. A circle of radius r is similar to a unit circle (a circle of radius 1), the similarity ratio being equal to r. (The number 1 here represents the unit of measure being used. It could be 1 inch, 1 foot, 1 meter, 1 centimeter, or any other convenient unit of distance. The radius r is measured relative to the same unit.)

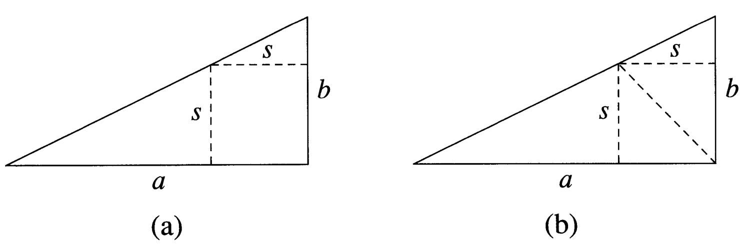
A circle, together with all the points inside the circle, is called a *circular disk*. If a unit circular disk has circumference C and area A, then a disk of radius r has circumference Cr (because lengths get multiplied by r) and area Ar^2 (because areas get multiplied by r^2). From the definition of the number π as the ratio of circumference to diameter we know that $C = 2\pi$. Archimedes made the remarkable discovery that $A = C/2$, so that $A = \pi$. Consequently, the area of a circular disk of radius r is πr^2. Methods of proving this formula are described in the program *The Story of* π.

EXERCISES ON AREAS OF TRIANGLES

1. This exercise shows how to deduce the formula for the area of a triangle from that of a rectangle. The rectangle in (a) has area ba. A right triangle of base b and altitude a shown in (a) fills half the rectangle so its area is $ba/2$. Use the diagrams in (b) or (c) to show that any triangle of base b and altitude a has area $ba/2$.

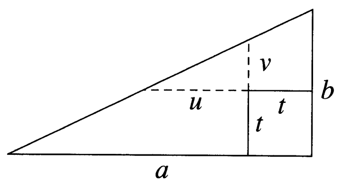


2. (a) A square of side s is inscribed in a right triangle with legs a and b as shown in (a). Use similar triangles to show that $s/(a - s) = (b - s)/s$. Solve this equation for s to obtain $s = ab/(a + b)$.

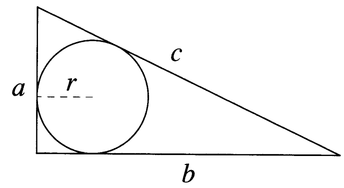


(b) Refer to the diagram in (b) and use an argument based on areas to find another derivation of the formula for s in part (a).

(c) Generalize the problem as follows. Draw a square of side t and let u and v be the lengths of the dotted lines shown in the figure below. Prove that $t = (ab - uv)/(a + b + u + v)$.



Exercise 2(c).



Exercise 3(a).

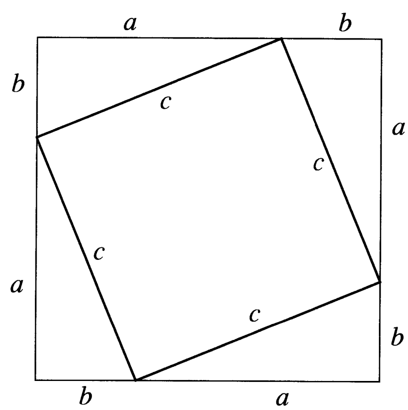
3. (a) A circle of radius r is inscribed in a right triangle with legs of length a and b and hypotenuse c, as shown above. Use an argument based on area to show that $r = ab/(a + b + c)$.

(b) Generalize the argument to show that if a, b, c are the sides of *any* triangle, then the radius r of the inscribed circle is given by $r = 2K/(a + b + c)$, where K is the area of the triangle.

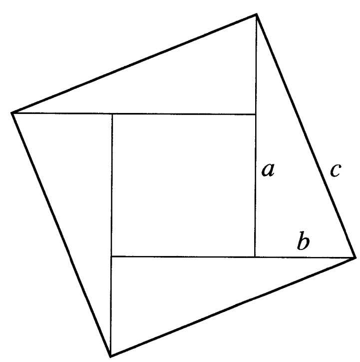
4. Early Chinese documents contain a proof of the Theorem of Pythagoras based on the diagram below. Essentially the same proof was discovered by U. S. Congressman James A. Garfield, a few years before he became the 20th president of the United States.

(a) Start with a square of edge $a + b$ and cut off four right triangles at the corners, each with hypotenuse c, and legs a and b, as shown in (a). Prove that the inner figure is a square.

(b) The area of the inner square, c^2, plus 4 times the area of each triangle, $ab/2$, is equal to the area of the larger square, $(a + b)^2$. Equate the areas to deduce $a^2 + b^2 = c^2$, the Theorem of Pythagoras.



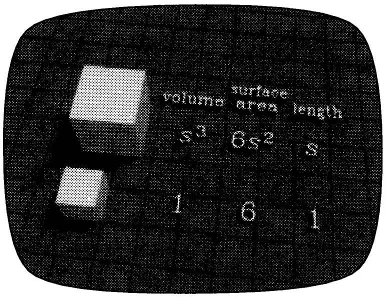
(a)

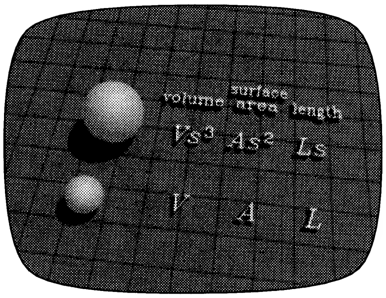


(c)

(c) In a similar vein, prove the Theorem of Pythagoras by placing the right triangles *inside* the square of edge c as shown in (c).

8. *Volumes of similar figures*





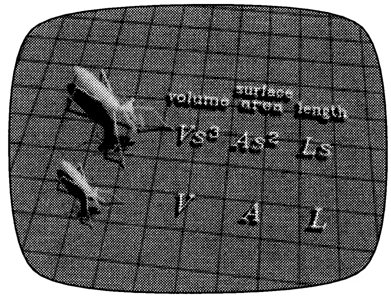


Figure 21 shows a unit cube (edge length 1), and a scaled copy with edge length 2. The larger cube consists of eight congruent copies of the original cube, as shown in Figure 21, so the volume is multiplied by 8, or 2^3, the cube of the scaling factor.

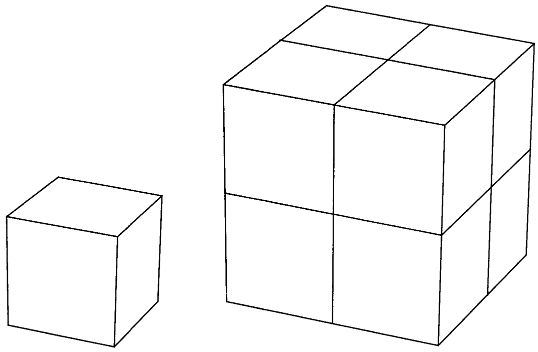


Figure 21. When the edges of a cube are multiplied by 2, its volume is multiplied by 2^3.

A discussion of volumes of general three-dimensional solids requires the methods of calculus. The underlying idea is fairly simple, and is somewhat analogous to our discussion of areas of plane regions. We fill the inside of the solid as closely as possible by a collection of cubes. The sum of the volumes of these interior cubes is taken as a lower approximation to the volume of the solid. We then add more cubes to the interior cubes until the new collection of cubes completely envelopes the entire solid. The sum of the volumes of all these cubes is taken as an upper approximation to the volume of the solid. If the inner and outer cubes are properly chosen and are made smaller and smaller, the lower and upper approximations can be made arbitrarily close to a certain number, and it is natural to call this number the volume of the solid. Scaling the solid by a factor s also scales the approximating cubes by the same factor, and their volumes are multiplied by the cube of the scaling factor. This idea leads to the following property of volumes of similar figures:

Scaling a solid figure by a factor s multiplies its volume by s^3.

Archimedes discovered that the volume of a sphere of radius r is $4\pi r^3/3$, which is r^3 times the volume of a unit sphere. He also found that its surface area is $4\pi r^2$. According to legend, Archimedes was so impressed by these discoveries that he wanted them recorded on his tombstone.

EXERCISES

1. By what factor does the weight of a fish increase if it grows from four inches long to a similar shape five inches long?

2. If a six-foot man weighs 160 pounds, calculate the weight of a scaled version of the same person of height (a) five feet; (b) four feet; (c) three feet; (d) eight feet; (e) ten feet; (f) twelve feet.

3. The radius of the moon is about 3/11 that of the earth. What is the ratio of the volume of the moon to that of the earth?

4. Given that a length of 1 meter is equal to 39.37 inches, or 3.281 feet, determine each of the following:

(a) 1 cubic meter expressed in cubic feet; in cubic inches.
(b) 1 cubic inch expressed in cubic meters.
(c) 1 cubic foot expressed in cubic meters.

5. What is the length of the edge of a cube whose volume is numerically equal to its surface area?

6. (a) If the radius of a sphere is increased by 10%, by what per cent is its volume increased?

(b) By what factor must the radius of a sphere be increased to double its volume?

7. Define the *density* of a material to be the ratio of its weight in grams (g) divided by its volume in cubic centimeters (cc). (This definition is used because one cc of pure water at temperature of maximum density is 1.000013 g, so the density of water is very nearly equal to 1 g/cc.) Determine the density of the following materials from the given data:

(a) Gold; 65 cc weighs 1251.77 g.
(b) Ice; 270 cubic meters weighs 229,000 kilograms.
(c) Pine wood; a cube 12 cm on each edge weighs 1 kilogram.

8. For an airplane of a given shape, assume that the minimum speed needed to keep it in the air is proportional to (i. e., is a constant times) the square root of its length; and the horsepower needed to maintain the minimum speed is proportional to the weight of the craft. If the linear dimensions are increased by a factor of four, determine:

(a) The factor the minimum speed must be multiplied by to keep the plane aloft.
(b) The factor the horsepower must be multiplied by to keep the plane aloft.

9. The famous astronomer Johannes Kepler discovered that, for any two planets, the cubes of their mean distances from the sun have the same ratio as the squares of their periods. (The period is the time it takes a planet to make one orbit around the sun.) The mean distance of the earth from the sun is called one astronomical unit (1 AU = 1.496×10^{11} meters), and its period is one year. Given that the mean distance of Jupiter from the sun in 5.20 astronomical units, find the period of Jupiter.

10. Refer to Exercise 9. The period of Saturn is 29.48 years. Calculate its mean distance from the sun (in astronomical units).

9. *Applications to biology*

Giants about sixty feet tall appear in classic literature such as "Jack and the Beanstalk," *Gulliver's Travels*, or *The Pilgrim's Progress*, and huge monsters such as King Kong, Godzilla, and Rodan are featured in science fiction movies. In a pioneering work, *On Growth and Form*, published in 1917, D'Arcy Thompson uses similarity to explain why such creatures cannot exist in our world. He quotes a passage that appeared some three hundred years earlier in Galileo's *Dialogues Concerning Two New Sciences*:

> ...if we tried building ships, palaces or temples of enormous size, yards, beams and bolts would cease to hold together; nor can Nature grow a tree nor construct an animal beyond a certain size, while retaining the proportions and employing the material which suffice in the case of a smaller structure. The thing will fall to pieces of its own weight unless we either change its relative proportions, which will at length cause it to become clumsy, monstrous and inefficient, or else we must find new material, harder and stronger than was used before.

A six-foot man scaled by a factor ten becomes a sixty-foot giant. His weight increases by a factor ten cubed, so he would weigh one thousand times as much, about eighty to ninety tons. But the strength of his bones increases like their cross sectional area, by a factor ten squared, so every square inch of giant bone would have to support ten times the weight borne by a square inch of human bone. The material in a human thigh bone would be crushed under this load, so these giants would collapse on broken legs after taking a single step. This also explains why fossils of animals the size of King Kong or Godzilla have never been found. *Tyrannosaurus rex*, a dinosaur resembling Godzilla, was the largest land-based predator that ever lived. It reached a height of about twenty feet, not much taller than a giraffe.

An object's ability to withstand pressure depends on the material it is made of. A sixty-foot statue made of steel or granite certainly would not collapse under its own weight. But even these materials are limited in their ability to resist the crushing force of gravity. A steel cube will crush under its own weight if each edge exceeds four miles. The maximum size for a granite cube is about seven miles on each edge. Therefore, a mountain made of granite could not be more than seven miles tall. Mt. Everest, the world's tallest peak, has a height of 29,002 feet, just under six miles. Galileo predicted that the tallest tree could not exceed 300 feet. And, in fact, the world's tallest trees, giant sequoias (unknown to Galileo), are about 360 feet tall. They exceed Galileo's limit because they have adapted their form to increase their strength beyond Galileo's model. The world's largest animals, blue whales, can exceed 70 feet in length and can weigh more than 100 tons. They are not crushed under their own weight because they are supported by the water in which they live.

Bone strength is not the only factor limiting the size of land animals. In a classic essay, "On Being the Right Size," J. B. S. Haldane explains that tall land animals have to pump blood to greater heights, requiring larger blood pressure and tougher blood vessels. Scaling by a factor ten multiplies weight a thousand times and, according to Haldane, the larger creature would need a thousand times as much food and oxygen per day and would excrete a thousand times as much of waste products. Because surface area is increased only a hundred times, ten times as much oxygen must enter per minute through each square inch of skin or lungs, and ten times as much food through each square inch of intestine. Lungs and intestines adapt by taking on more convoluted shapes in the struggle to increase surface area in proportion to volume. The same is true of plants. Simple plants such as green algae are round cells. Higher plants increase their surface area by putting out leaves and roots.

So there are several arguments, based on similarity, explaining why a giant creature such as a praying mantis the size of a horse cannot exist in the real world. But such creatures do exist in the fantasy world of literature, movies and television. The clever use of mechanical scale models and special effects makes a movie giant like King Kong look real. Ingenious photography of small creatures such as insects can make them look like giants, as for example the Manhole Monsters in the video program *Similarity*.

Haldane's essay also explains that the force of gravity, which limits the size of land animals, presents no danger to small creatures:

> You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away. A rat is killed, a man is broken, a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.
>
> An insect, therefore, is not afraid of gravity; it can fall without danger, and can cling to the ceiling with remarkably little trouble. It can go in for elegant and fantastic forms of support like that of the daddy-long-legs. But there is a force which is as formidable to an insect as gravitation to a mammal. This is surface tension. A man coming out of a bath carries with him a film of water of about one-fiftieth of an inch in thickness. This weighs roughly a pound. A wet mouse has to carry about its own weight of water. A wet fly has to lift many times its own weight and, as everyone knows, a fly once wetted by water or any other liquid is in a very serious position indeed. An insect going for a drink is in as great danger as a man leaning out over a precipice in search of food. If it once falls into the grip of the surface tension of the water--that is to say, gets wet--it is likely to remain so until it drowns. A few insects, such as water-beetles, contrive to be unwettable; the majority keep well away from their drink by means of a long proboscis.

Similarity helps explain why a hummingbird's heart beats about sixteen times faster than a human heart. A hummingbird four inches long is about one-sixteenth the average height of a person, but the amount of blood in its body is proportional to its volume, which is about one-sixteenth cubed of a person's volume. The surface area, through which body heat escapes, is about one-sixteenth squared of a person's surface area. So the bird has a surface area about one-sixteenth squared of ours, but only one-sixteenth cubed as much blood to keep it warm. Therefore, its heart has to pump about sixteen times as fast. That's roughly a thousand beats per minute. Of course, the heart rate of a hummingbird (and of a human) varies with its activity and depends on other factors as well, so these numbers must be considered as rough estimates. But the order of magnitude is consistent with observed values. In their book *Hummingbirds, Their Life and Behavior*, Esther Quesada Tyrell and Robert A. Tyrrell report that the heart rate of a hummingbird varies from approximately 500 times per minute when it is at rest, to more than 1200 times per minute when it is excited.

EXERCISES

1. The gravitational force on the moon is about 1/6 that on earth. If the maximum height of a mountain on earth is seven miles, what is the maximum height on the moon?

2. Explain why a whale 16 times the length of a human can hold its breath under water 16 times as long.

3. How would you design an experiment to determine the size of the largest cube of Jello that will not collapse under its own weight?

10. *Recap*

In this program we've seen that scaling a figure does not change its shape. So, if you know something about one shape, then you also know something about every similar shape. For any two similar polygons, corresponding angles are equal, lengths of corresponding line segments have the same ratios, and corresponding internal ratios are the same. Line segments and perimeters are multiplied by the scaling factor, areas by the square of the scaling factor, and volumes by the cube of the scaling factor.

The concept of similarity is one of the great triumphs of Euclidean geometry, with applications extending far beyond geometry. Similarity plays an important role in every aspect of art, science or technology involving measurement. It reveals the secret of map making, scale drawings and blueprints, and also explains some aspects of photographic images and vision itself. Similarity also plays a role in regulating the size and structure of life forms. For every type of plant or animal there seems to be an optimum size, and similarity helps explain why a large change in size always carries with it a change of form.

Suggested references for further study

1. Solomon Garfunkel, *For All Practical Purposes,* W. H. Freeman and Co., 1988.

2. J. B. S. Haldane, *On Being the Right Size and Other Essays,* Oxford University Press, 1985.

3. Thomas A. McMahon, John Tyler Bonner, *On Size and Life,* Scientific American Books, Inc., 1983.

4. D'Arcy Wentworth Thompson, *On Growth and Form,* Cambridge University Press, 1917.

5. Esther Quesada Tyrrell, Robert A. Tyrrell, *Hummingbirds, Their Life and Behavior.* Crown Publishers, 1985.

INDEX

Project MATHEMATICS!

COMPUTER ANIMATED MATHEMATICS VIDEOTAPES

Similarity is part of a series of modules that use computer animation to help instructors teach basic concepts in mathematics. Each module consists of a videotape, about 20 minutes in length, and a workbook to guide the students through the video, elaborating on the important ideas. The modules are used as support material for existing courses in high school and community college classrooms, and may be copied without charge for educational use.

Based at the California Institute of Technology, *Project MATHEMATICS!* has attracted as partners the departments of education of 36 states in a consortium whose members reproduce and distribute the videotapes and written materials to public schools. The project is headed by Tom M. Apostol, professor of mathematics at Caltech and an internationally known author of mathematics textbooks. Co-director of the project is James F. Blinn, one of the world's leading computer animators, who is well known for his Voyager planetary flyby simulations. Blinn and Apostol worked together previously as members of the academic team that produced *The Mechanical Universe*, an award-winning physics course for television.

The first six videotapes produced by *Project MATHEMATICS!* have been distributed to thousands of classrooms nationwide. *The Theorem of Pythagoras* has received first-place awards at many international competitions, including a Gold Cindy at the 1989 Cindy Competition, Los Angeles, California. *The Story of Pi* was awarded a Gold Apple Award at the 1990 National Educational Film & Video Festival in Oakland, and a Red Ribbon Award at the 1990 American Film and Video Festival. *Similarity* was awarded a Silver Apple at the 1991 National Educational Film and Video Festival in Oakland, California. *Sines and Cosines, Part I*, captured a silver medal at the 1993 New York Film and Video Festival.

Information about the project can be obtained by writing to the project director at the address on the title page of this booklet. Copies of videotapes and workbooks for *The Theorem of Pythagoras, The Story of Pi, Similarity, Polynomials, The Teachers Workshop, Sines and Cosines, Part I,* and *Sines and Cosines, Part II* can be obtained at nominal charge from the Caltech Bookstore, 1-51 Caltech, Pasadena, CA 91125. (Telephone: 818-356-6161 until 7-1-93; new number 818-395-6161 after 7-1-93) Most of these titles can also be obtained from the National Council of Teachers of Mathematics, and the Mathematical Association of America at the addresses given below.

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